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1991 J. Phys. A: Math. Gen. 24 407

(http://iopscience.iop.org/0305-4470/24/2/014)

J. Phys. A: Math. Gen. 24 (1991) 407-413. Printed in the UK

Path integral quantization of non-relativistic systems with magnetic charges

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Received 26 July 1990

Abstract. The exact path integration for systems with magnetic charges is presented. The Green function and bound-state energy spectrum of a dyonium are obtained. Unlike earlier works, the exact path integration is done directly in spherical polar coordinates and does not require the use of coordinate transformations such as the non-bijective transformation of Kustaanheimo and Stiefel.

1. Introduction

In this paper, we demonstrate the direct evaluation of the path integral in spherical polar coordinates for non-relativistic systems with magnetic charges [1-3]. The method we give sets a natural framework for achieving the separation of the propagator and the Green function into a radial part and the Wu-Yang monopole harmonics [4]. This is in contrast to earlier path integral approaches for Coulomb problems involving magnetic charges such as that of Dürr and Inomata [5] which makes use of parabolic rotational coordinates, and that of Kleinert [6] which applies the R^4 to R^3 non-bijective Kustaanheimo-Stiefel (KS) mapping [7]. Note that the KS coordinates in four-dimensional space are directly related to parabolic rotational coordinates when parametrized in terms of Euler angles. Hence, the monopole harmonics of Wu and Yang are recovered using the KS approach when the transformation to angular variables is carried out. As we shall see, however, there is no need to go through these coordinate transformations because we can readily obtain the separated Green function or the propagator by directly evaluating the path integral in spherical polar coordinates. The evaluation of the radial part is then subject to the specification of the scalar interaction potential $V(r)$.

To solve problems involving magnetic charges directly in spherical polar coordinates, we shall see that the procedure for path integrating a class of non-central potentials involving both the polar and azimuthal angles recently presented [8]† proves essential. Here, the repeated application of Bateman's expansion formula [9] within the path integral allows the complete separation of the propagator or the Green function into angular and radial parts.

In this paper, we shall focus our solution on dyon systems [10-12]. In particular, the bound-state energy spectrum is obtained for the dyonium. Naturally included in

† See Khandekar and Lawande in [8] for a class of potentials of the form $V(r, \theta)$ and Peak and Inomata in [8] for central potentials. Path integration in polar coordinates was treated first by Edwards and Gulyaev.

the solution presented are the special cases of an electrically charged particle moving around a magnetic monopole or around a dyon.

2. The propagator for systems with magnetic charges

We shall consider a light dyon of charge (e_1, g_1) interacting with a heavy dyon of charge (e_2, g_2) located at the origin and with a central scalar potential $V(r)$. The e and g refer to the electric and magnetic charges, respectively. The Lagrangian for this dyon–dyon system is given by

$$L=\tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2+qA(\boldsymbol{r})\cdot\dot{\boldsymbol{r}}-V(r) \tag{2.1}$$

where μ is the reduced mass and $q=e_1g_2-e_2g_1$. Special cases include the charge–dyon interaction $(g_1=0)$, the charge–monopole system $(g_1=e_2=0)$ and, of course, the Coulomb problem $(g_1=g_2=0)$.

Following Wu and Yang [4], the vector potential $A(\boldsymbol{r})$ is chosen to be

$$(A_r)_{\mathrm{a}}=(A_\theta)_{\mathrm{a}}=0 \qquad (A_\phi)_{\mathrm{a}}=q(1-\cos\theta)(r\sin\theta)^{-1} \tag{2.2a}$$

$$(A_r)_{\mathrm{b}}=(A_\theta)_{\mathrm{b}}=0 \qquad (A_\phi)_{\mathrm{b}}=-q(1+\cos\theta)(r\sin\theta)^{-1} \tag{2.2b}$$

valid for the overlapping regions $R_{\mathrm{a}}=\{0\leqslant\theta<\frac{1}{2}\pi+\delta\}$ and $R_{\mathrm{b}}=\{\frac{1}{2}\pi-\delta<\theta\leqslant\pi\}$, respectively, with $0<\delta\leqslant\frac{1}{2}\pi$. We simply note the fibre bundle structure of the resulting wavefunctions and concentrate on the solutions analytic in a specified region, i.e. ψ_{a} and ψ_{b} corresponding to R_{a} and R_{b}, respectively.

The propagator for the particle of mass μ described by the Lagrangian (2.1) can be written in terms of the path integral [13],

$$K(\boldsymbol{r}'',\boldsymbol{r}',\tau)=\int\exp\left(\frac{\mathrm{i}}{\hbar}S\right)\mathscr{D}(\boldsymbol{r}) \tag{2.3}$$

where $S=\int_{t'}^{t''}L\,\mathrm{d}t$ is the action and $\tau=t''-t'$. In time-sliced form, the propagator is given by

$$K(\boldsymbol{r}'',\boldsymbol{r}';\tau)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}}{\hbar}S_j\right)\left(\frac{\mu}{2\pi\mathrm{i}\hbar\tau_j}\right)^{3/2}\prod_{j=1}^{N-1}\mathrm{d}^3\boldsymbol{r}_j \tag{2.4}$$

where the action for each time subinterval, $\tau_j=t_j-t_{j-1}$, has the form

$$S_j=\frac{\mu}{2\tau_j}(r_j^2+r_{j-1}^2)-\frac{\mu}{\tau_j}r_jr_{j-1}\cos\Theta_j+qA(\boldsymbol{r}_j)\cdot\dot{\boldsymbol{r}}_j\tau_j-V(r_j)\tau_j \tag{2.5}$$

where

$$\cos\Theta_j=\cos\theta_j\cos\theta_{j-1}+\sin\theta_j\sin\theta_{j-1}\cos(\phi_j-\phi_{j-1}). \tag{2.6}$$

The velocity-dependent term in (2.5) must be evaluated with care. End-point evaluations of the vector potential, i.e. $A(\boldsymbol{r}_{j+1})$ and $A(\boldsymbol{r}_j)$, give rise to non-negligible differences in the contributions to the action integral. Following Feynman [14], we take the arithmetic average, $\frac{1}{2}[A(\boldsymbol{r}_{j+1})+A(\boldsymbol{r}_j)]\cdot(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)$. For the case of the monopole potential (2.2), the contribution of the velocity-dependent potential is just $q(\pm1-\cos\bar{\theta}_j)\Delta\phi_j+\mathrm{O}(\tau_j^{3/2})$ in which $\bar{\theta}_j=(\theta_j+\theta_{j-1})/2$ and $\Delta\phi_j=\phi_j-\phi_{j-1}$. With this, (2.5) can be written as

$$S_j = \frac{\mu}{2\tau_j}(r_j^2 + r_{j-1}^2) - \frac{\mu}{\tau_j} r_j r_{j-1} \cos \Theta_j + q(\pm 1 - \cos \bar{\theta}_j)\Delta\phi_j - V(r_j)\tau_j \tag{2.7}$$

since terms of $O(\tau_j^{1+\varepsilon})$, for $\varepsilon > 0$, can be ignored in path integration. With (2.4) and (2.7), a path integral evaluation is carried out in the following sections.

3. Evaluating the angular path integrals

There are two stages involved in performing the angular path integrations in (2.4) with the action (2.7). The first stage is the separation of the ϕ-dependent terms. The second entails carrying out the separation of the θ-part from the radial part. Once a complete separation is achieved, orthogonality relations of the partial angular wavefunctions allow the multiple angular integrals to be evaluated in closed form. These yield nothing but the monopole harmonics.

The ϕ-coordinate can be separated from the other variables by utilizing the asymptotic expansion for the modified Bessel functions $I_\nu(z)$, for any assigned value of ν and large (complex) z, valid for $|\arg(u/\tau_j)| < \pi/2$,

$$I_\nu(u/\tau_j) \approx (2\pi u/\tau_j)^{-1/2} \exp\left(\frac{u}{\tau_j} - (\nu^2 - \tfrac{1}{4})\frac{\tau_j}{2u} + O(\tau_j^2)\right). \tag{3.1}$$

Here we have dropped the negligible second series in the expansion [15] of $I_\nu(z)$. With (3.1), the following relation is obtained [16]†:

$$\exp\left[\frac{u}{\tau_j}\cos\left(\Delta\phi + \frac{\mathrm{i}y\tau_j}{u}\right) - \frac{y^2\tau_j}{2u}\right] \approx \sum_{m=-\infty}^{\infty} \exp(\mathrm{i}m\Delta\phi) I_{m+y}(u/\tau_j). \tag{3.2}$$

The ϕ-dependent terms in the propagator can be written in the same form as the left-hand side of (3.2) by noting that for small τ_j we have the relation $\cos(\Delta\phi_j) - a\tau_j\Delta\phi_j \approx \cos(\Delta\phi_j + a\tau_j) + a^2\tau_j^2/2$. Application of (3.2), then makes the path integration for the ϕ-coordinate in (2.7) straightforward, yielding

$$\begin{aligned} K(\mathbf{r}'', \mathbf{r}'; \tau) = \lim_{N\to\infty} (2\pi)^{N-1} & \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}(m \pm q/\hbar)(\phi'' - \phi')] \\ & \times \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}\mu}{2\hbar\tau_j}(r_j^2 + r_{j-1}^2) - \frac{\mathrm{i}\mu}{\hbar\tau_j} r_j r_{j-1} \cos\theta_j \cos\theta_{j-1} - \mathrm{i}V(r_j)\tau_j/\hbar\right) \\ & \times I_{m+y}(\mu r_j r_{j-1} \sin\theta_j \sin\theta_{j-1}/\mathrm{i}\hbar\tau_j) \prod_{j=1}^{N} (\mu/2\pi\mathrm{i}\hbar\tau_j)^{3/2} \\ & \times \prod_{j=1}^{N-1} (r_j^2\, \mathrm{d}r_j \sin\theta_j\, \mathrm{d}\theta_j). \end{aligned} \tag{3.3}$$

† Note that this work by Bernido and Inomata, which treats a particle in an Aharonov–Bohm potential field, provided an early example with non-trivial dependence on an angular variable that can be handled exactly. A treatment of an extended class is given by Cheng and De Souza Dutra in [16], which also entails evaluation of the constrained path integral found in Edwards [16].

To separate the radial coordinate from the θ-variable, we proceed by making use of (3.1), allowing us to write (3.3) as

$$K(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \lim_{N\to\infty} (2\pi)^{N-1} \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}(m \pm q/\hbar)(\phi'' - \phi')]$$
$$\times \int \prod_{j=1}^{N} (2\pi\mu r_j r_{j-1} \sin\theta_j \sin\theta_{j-1}/\mathrm{i}\hbar\tau_j)^{-1/2}$$
$$\times \exp\left(\frac{\mathrm{i}\mu}{2\hbar\tau_j}(r_j^2 + r_{j-1}^2) - \mathrm{i}V(r_j)\tau_j/\hbar + \frac{\mu}{\mathrm{i}\hbar\tau_j} r_j r_{j-1} \cos(\Delta\theta_j)\right.$$
$$-\mathrm{i}\tfrac{1}{2}[(m+q/\hbar)^2 - \tfrac{1}{4}]\hbar\tau_j/4\mu r_j r_{j-1} \sin(\theta_j/2)\sin(\theta_{j-1}/2)$$
$$-\mathrm{i}\tfrac{1}{2}[(m-q/\hbar)^2 - \tfrac{1}{4}]\hbar\tau_j/4\mu r_j r_{j-1} \cos(\theta_j/2)\cos(\theta_{j-1}/2)$$
$$\left. + \mathrm{i}q^2\tau_j/2\hbar\mu r_j r_{j-1}\right) \prod_{j=1}^{N} (\mu/2\pi\mathrm{i}\hbar\tau_j)^{3/2} \prod_{j=1}^{N-1} (r_j^2\, \mathrm{d}r_j \sin\theta_j\, \mathrm{d}\theta_j). \tag{3.4}$$

With the help of the relation $\cos(\Delta\theta_j) = 4\cos(\Delta\theta_j/2) + (\Delta\theta_j/2)^4/2 - 3 + \mathrm{O}(\tau_j^2)$, together with the designation of ξ_1 and ξ_2 as

$$\xi_1 = \frac{4\mu}{\mathrm{i}\hbar} r_j r_{j-1} \sin(\theta_j/2)\sin(\theta_{j-1}/2) \tag{3.5}$$

$$\xi_2 = \frac{4\mu}{\mathrm{i}\hbar} r_j r_{j-1} \cos(\theta_j/2)\cos(\theta_{j-1}/2) \tag{3.6}$$

equation (3.4) acquires the form

$$K(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \lim_{N\to\infty} (2\pi)^{N-1} \sum_{m=-\infty}^{\infty} \exp[\mathrm{i}(m \pm q/\hbar)(\phi'' - \phi')]$$
$$\times \int \prod_{j=1}^{N} (\mathrm{i}\hbar\tau_j/8\pi\mu r_j r_{j-1})^{-1/2} \exp\left(\frac{\mathrm{i}\mu}{2\hbar\tau_j}(r_j^2 + r_{j-1}^2)\right.$$
$$- \mathrm{i}V(r_j)\tau_j/\hbar + [(4q/\hbar)^2 + 3]\mathrm{i}\hbar\tau_j/32\mu r_j r_{j-1}$$
$$\left. - (3\mu r_j r_{j-1}/\mathrm{i}\hbar\tau_j)\right) I_\alpha(\xi_1/\tau_j) I_\beta(\xi_2/\tau_j)$$
$$\times \prod_{j=1}^{N} (\mu/2\pi\mathrm{i}\hbar\tau_j)^{3/2} \prod_{j=1}^{N-1} (r_j^2\, \mathrm{d}r_j \sin\theta_j\, \mathrm{d}\theta_j). \tag{3.7}$$

Here, $\alpha = m + q/\hbar$ and $\beta = m - q/\hbar$. Up to this point, the θ and r variables are still mixed. Now, we make use of Bateman's expansion formula [9],

$$\tfrac{1}{2}\zeta I_\alpha[\zeta \sin(\theta_j/2)\sin(\theta_{j-1}/2)] I_\beta[\zeta\cos(\theta_j/2)\cos(\theta_{j-1}/2)]$$
$$= [\sin(\theta_j/2)\sin(\theta_{j-1}/2)]^\alpha [\cos(\theta_j/2)\cos(\theta_{j-1}/2)]^\beta$$
$$\times \sum_{l=0}^{\infty} \left((\alpha+\beta+2l+1) I_{\alpha+\beta+2l+1}(\zeta) \frac{l!\,\Gamma(\alpha+\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+l+1)}\right.$$
$$\left. \times P_l^{\alpha,\beta}(\cos\theta_j) P_l^{\alpha,\beta}(\cos\theta_{j-1})\right) \tag{3.8}$$

which effectively separates the r and θ coordinates with $\zeta = 4\mu r_j r_{j-1}/\mathrm{i}\hbar\tau_j$ and $P_l^{\alpha,\beta}$ the Jacobi polynomials. The orthogonality condition for the Jacobi polynomials facilitates the integration of the θ-variable allowing us to write equation (3.7) as

$$K(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \sum_{l=q/\hbar}^{\infty} \sum_{m=-l}^{l} R_l(r'', r'; \tau) \exp[\mathrm{i}(m \pm q/\hbar)(\phi'' - \phi')]$$
$$\times [\cos(\theta''/2)\cos(\theta'/2)]^{\beta}[\sin(\theta''/2)\sin(\theta'/2)]^{\alpha}$$
$$\times \frac{(\alpha+\beta+2l+1)l!\,\Gamma(\alpha+\beta+l+1)}{4\pi\Gamma(\alpha+l+1)\Gamma(\beta+l+1)} P_l^{\alpha\beta}(\cos\theta')P_l^{\alpha,\beta}(\cos\theta''). \tag{3.9}$$

Equation (3.9) can be written as $K(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \Sigma R_l(r'', r'; \tau)\mathcal{Y}(\theta'', \phi'')\mathcal{Y}(\theta', \phi')$, where the angular functions are the monopole harmonics of Wu and Yang [4],

$$\mathcal{Y}_{q/\hbar,l,m}(\theta, \phi)$$
$$= 2^{-m}\left(\frac{(2l+1)(l-m)!(l+m)!}{4\pi(l+q/\hbar)!(l-q/\hbar)!}\right)^{1/2}(1-x)^{\alpha/2}(1+x)^{\beta/2}$$
$$\times P_l^{\alpha,\beta}(x)\exp[\mathrm{i}(m \pm q/\hbar)\phi] \tag{3.10}$$

with $x = \cos\theta$, $\alpha = m + q/\hbar$ and $\beta = m - q/\hbar$. Our normalization factor differs with that of Wu and Yang by a factor 2^{2m}.

The radial part which remains to be path integrated once $V(r)$ is specified is of the form

$$R_l(r'', r'; \tau) = \lim_{N\to\infty} (4\pi)^N \int \prod_{j=1}^{N} (2\pi\mathrm{i}\hbar\tau_j/\mu r_j r_{j-1})^{1/2}$$
$$\times \exp\left(\frac{\mathrm{i}\mu}{2\hbar\tau_j}(r_j^2 + r_{j-1}^2) - \mathrm{i}V(r_j)\tau_j/\hbar + \frac{[(4q/\hbar)^2+3]\mathrm{i}\hbar\tau_j}{32\mu r_j r_{j-1}} - \frac{3\mu}{\mathrm{i}\hbar\tau_j} r_j r_{j-1}\right)$$
$$\times I_{2m+2l+1}(4\mu r_j r_{j-1}/\mathrm{i}\hbar\tau_j)\prod_{j=1}^{N}(\mu/2\pi\mathrm{i}\hbar\tau_j)^{3/2}\prod_{j=1}^{N-1}(r_j^2\,\mathrm{d}r_j). \tag{3.11}$$

4. Bound-state spectrum for the dyonium

In tackling the dyonium for which $V(r) = -\kappa/r$, where $\kappa = -(e_1e_2 + g_1g_2)$, we note that, for systems with Coulombic radial dependence, the propagator has not yet been evaluated exactly. Instead, the Green function is evaluated since its poles and residues at the poles provide the discrete energy spectrum and the eigenfunctions, respectively [17–19]. The Green function can be obtained from the Fourier transform of the propagator $K(\boldsymbol{r}', \boldsymbol{r}''; \tau)$,

$$G(\boldsymbol{r}'', \boldsymbol{r}'; E) = \frac{1}{\mathrm{i}\hbar}\int K(\boldsymbol{r}'', \boldsymbol{r}'\tau)\exp\left(\frac{\mathrm{i}}{\hbar}E\tau\right)\mathrm{d}\tau$$
$$= \frac{1}{\mathrm{i}\hbar}\int P(\boldsymbol{r}'', \boldsymbol{r}'; \tau)\,\mathrm{d}\tau. \tag{4.1}$$

In a manner similar to that for the path integral representation of the propagator, the promotor $P(\boldsymbol{r}'', \boldsymbol{r}'; \tau)$ in (4.1) can be expressed as a path integral in terms of a modified action $W = \int (L + E)\,\mathrm{d}t$. For the case of the dyonium potential, the procedure in section

3 can be applied to separate the monopole harmonics from the radial part leading to a promotor of the form, $P(r'', r'; \tau) = \Sigma\, P_l(r'', r'; \tau)\mathscr{Y}(\theta'', \phi'')\mathscr{Y}(\theta', \phi')$. The radial part is given by

$$P_l(r'', r'; \tau) = \lim_{N\to\infty} (r_0 r_N)^{-1} \int \prod_{j=1}^{N} \exp(\mathrm{i}\, W_j/\hbar) \prod_{j=1}^{N} (\mu/2\pi \mathrm{i}\hbar\tau_j)^{1/2} \prod_{j=1}^{N-1} (\mathrm{d}r_j) \tag{4.2}$$

where the effective short-time action is

$$W_j = \frac{\mu}{2\tau_j}(r_j^2 + r_{j-1}^2) + \frac{\kappa\tau_j}{r_j} + \frac{[(4q/\hbar)^2 - 4(2l+1)^2 + 4]\hbar^2\tau_j}{32\mu r_j r_{j-1}} - \frac{\mu}{\tau_j} r_j r_{j-1} + E\tau_j. \tag{4.3}$$

The path integration with this short-time action can be done following the procedure given by Inomata in his evaluation of the radial path integral for the Coulomb problem [19]. This yields the radial Green function,

$$\begin{aligned} G_l(r'', r'; E) &= (\mathrm{i}\hbar)^{-1} \int P_l(r'', r'; \tau)\, \mathrm{d}\tau \\ &= (2\mu/\hbar^2)(2\mathrm{i}kr''r')^{-1} \frac{\Gamma(p+\gamma+1)}{\Gamma(2\gamma+2)} M_{-p,\gamma+(1/2)}(-2\mathrm{i}kr') \\ &\quad \times W_{-p,\gamma+(1/2)}(-2\mathrm{i}kr'') \end{aligned} \tag{4.4}$$

where $M(-2\mathrm{i}kr')$ and $W(-2\mathrm{i}kr'')$ are Whittaker functions. Here $k = (2\mu E)^{1/2}/\hbar$, $p = \mathrm{i}(\mu\kappa^2/2\hbar^2 E)^{1/2}$ and $\gamma = [(l+\frac{1}{2})^2 - (q/\hbar)^2]^{1/2} - \frac{1}{2}$.

The discrete energy spectrum for the dyon-dyon interaction can be obtained from the poles of the gamma function $\Gamma(z)$, i.e. when $p+\gamma+1 = -n_r (n_r = 0, 1, 2, \ldots)$. This yields

$$E_n = -(m\kappa^2/2\hbar^2 n^2) \qquad n = n_r + \gamma + 1. \tag{4.5}$$

Equations (4.4) and (4.5) coincide with the previous results for the dyonium [5, 6, 11]. By taking $e_2 = g_1 = 0$, we can recover the results for the charge-monopole interaction [2-4]. Naturally, when the magnetic charges are $g_1 = g_2 = 0$, the results reduce to the Coulomb wavefunctions and spectrum [17-19].

5. Conclusion

The direct evaluation of the path integral in spherical polar coordinates for the dyon-dyon system, with its special cases such as the charge-monopole and Coulomb systems, as presented above, provides an alternative to the treatment using parabolic coordinates [5] or the Kustaanheimo-Stiefel transformation [6]. The steps used in the preceding sections are straightforward and may be applied to provide another way of treating magnetic charges in higher-dimensional theories, such as the Kaluza-Klein monopole [20].

Acknowledgments

The authors would like to thank Professor A Inomata for helpful discussions. This research was partially supported by the National Research Council of the Philippines.

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